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B_n Stanley symmetric functions

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Abstract

We use the Kraśkiewicz insertion algorithm to show that the B_n Stanley symmetric function can be expressed as a nonnegative integer combination of Schur P-functions. It turns out that the decreasing parts of the insertion tableau form a shifted tableau, and Edelman–Greene insertion can be regarded as a special case of Kraśkiewicz insertion. Using these results, we obtain simple forms for some of the B_n Stanley symmetric functions. We show that the recording tableau of a reduced word $\boldsymbol{a}$ under Kraśkiewicz insertion is the evacuation of the recording tableau of the reverse of $\boldsymbol{a}$. We also establish a connection between Kraśkiewicz insertion and promotion sequence.

Résumé

Nous employons l'algorithme d'insertion de Kraśkiewicz afin de démontrer que la fonction symétrique B_n de Stanley peut être exprimée comme une combinaison de P-fonctions de Schur à coefficients positifs. Il s'avère que les parties décroissantes du tableau d'insertion forment un tableau gauche, et l'insertion d'Edelman–Greene peut être perçue comme un cas spécial de l'insertion de Kraśkiewicz. Par le biais de ces resultats, nous obtenons des formes simples pour certaines des fonctions symétriques B_n de Stanley. Nous montrons que le 'recording tableau' d'un mot réduit a par rapport à l'insertion de Kraśkiewicz est l'évacuation du 'recording tableau' de l'image-miroir de a. Nous établissons également une 'relation entre l'insertion de Kraśkiewicz et la suite de promotion.

1. Introduction

In [4], Edelman and Greene gave an insertion algorithm that present a bijection between the reduced words of elements of the symmetric group S_n and the set $\{(P, Q)$: P is a tableau word and Q is a standard Young tableau$\}$. This gives rise to the formula

$$F_w(x) = \sum s_\lambda(x), \tag{1}$$

where $F_w(x)$ is the symmetric function appearing in [17]. The sum is over all the insertion tableaux that can be obtained from reduced words of w. This symmetric function was used by Stanley to enumerate the number of reduced words of a given

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permutation. It turns out that this function is also a stable Schubert polynomial [1, 3, 12]. An analogue of the insertion algorithm for the hyperoctahedral group B_n was developed by Kraśkiewicz in [10]. In this paper, we use Kraśkiewicz insertion to show that the B_n Stanley symmetric function $G_w(x)$ can be expressed as a non-negative sum of Schur P-functions. Thus, we obtain a formula analogous to (1). We also find B_n analogues of some results in [4].

2. B_n Stanley symmetric function

We take our notations and nomenclature for the hyperoctahedral group B_n (also called the group of signed permutations) from [9]. We write an element of B_n in 1-line notation. So $\bar{3}421$ represents the signed permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \bar{3} & 4 & 2 & 1 \end{pmatrix}.$$

From [9], B_n is generated by the simple reflections

$$s_0 = \bar{1}2 \cdots n,$$

$$s_i = 12 \cdots i-1\, i+1\, i\, i+2 \cdots n \quad \text{for } 1 \leqslant i \leqslant n.$$

This means that any signed permutation w can be expressed as a product of these simple reflections. Any such expression for w of shortest length is called a *reduced word* (or *reduced decomposition*) of w. We denote this shortest length by $l(w)$ and the collection of reduced words of w by $R(w)$. Also, we denote the number of bars in the 1-line notation of w by $l_0(w)$. Often, for convenience, we write only the subscripts of the simple reflections in a reduced word. For example, $s_3s_2s_1s_0s_3s_2s_3$ may be written as 3210323.

We now define the B_n Stanley symmetric functions using compatible sequences. There are other equivalent definitions; see [5, 2]. The definition of compatible sequences imitates the formulation of Schubert polynomial and stable Schubert polynomials in [17, Eq. (1); 13, p. 101; 3, Theorem 1.1; 6].

We begin with the definition of a unimodal sequence.

Definition 2.1. A sequence of integers $\boldsymbol{a} = a_1a_2 \cdots a_l$ is called *unimodal* if there exists k such that

$$a_1 > a_2 > \cdots > a_k < a_{k+1} < \cdots < a_l.$$

We call $\boldsymbol{a}\downarrow = a_1a_2 \cdots a_k$ the *decreasing part* of $\boldsymbol{a}$ and $\boldsymbol{a}\uparrow = a_{k+1}a_{k+2} \cdots a_l$ the increasing part of $\boldsymbol{a}$.

Note that a unimodal sequence always has a decreasing part.

To define a compatible sequence, we consider two different types of numbers, barred and unbarred, with the linear order

$$\bar{1} < 1 < \bar{2} < 2 < \bar{3} < \cdots.$$

Definition 2.2. Let $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$. We say that a sequence of barred and unbarred positive intergers $\boldsymbol{i} = i_1 i_2 \cdots i_m$ is an $\boldsymbol{a}$-compatible sequence if

1. $i_i \leqslant i_2 \leqslant \cdots \leqslant i_m$,
2. $i_i = i_{j+1} = \cdots = i_k = \bar{l}$ occurs only when $a_j > a_{j+1} > \cdots > a_k$,
3. $i_j = i_{j+1} = \cdots = i_k = l$ occurs only when $a_j < a_{j+1} < \cdots < a_k$, and
4. i_j is unbarred if $a_j = 0$.

Denote the set of $\boldsymbol{a}$-compatible sequences by $K(\boldsymbol{a})$.

Definition 2.3. Let $w \in B_n$, define the B_n *Stanley symmetric function* of w as follows:

$$G_w(x) = \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{i} \in K(\boldsymbol{a})} x_{|i_1|} x_{|i_2|} \cdots x_{|i_m|},$$

where $|i_j| = l$ if $i_j = \bar{l}$ or l.

So $G_w(\boldsymbol{x})$ is a formal power series of degree $l(w)$ in the variables $x_1, x_2, \cdots$. This definition arises from algebraic considerations (see [5, 2]). It is not obvious that $G_w(\boldsymbol{x})$ is a symmetric function. However, we refer the reader to [5] where this fact follows easily from an alternative definition. We quote the result here.

Theorem 2.4 (Fomin and Kirillov [5]). *For all $w \in B_n$, $G_w(x)$ is a symmetric function and can be expressed as an integer combination of Schur P-functions.*

The theory of Schur P-functions parallels that of Schur functions [19, 14]. We now define Schur P-functions in terms of shifted tableaux.

A *shifted tableau* T is a collection of boxes filled with numbers, arranged into l rows of strictly decreasing lengths $\lambda_1, \lambda_2, \ldots, \lambda_l$. Each row is indented one box to the right of the row above. The partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is called the *shape* of the shifted tableau and is denoted $\mathrm{sh}(T)$. We do not assume any conditions about the order of the elements within the rows and columns.

Example

<table>
<tr><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td></td><td>5</td><td>6</td><td>7</td></tr>
</table>

is a shifted tableau of shape (4, 3).

Definition 2.5. A shifted tableau T with n boxes is called *a standard shifted Young Tableau if*

1. the entries are all distinct ranging from 1 to n,
2. the numbers in each row is strictly increasing, and
3. the numbers in each column is strictly increasing.

Definition 2.6. A shifted tableau T filled with barred and unbarred numbers is called a *semistandard shifted Young tableau* if

1. all the numbers are weakly increasing along each row and each column,
2. the unbarred numbers are strictly increasing in each column, and
3. the barred numbers are strictly increasing in each row.

To each semistandard shifted Young tableau T of size n, we associate a monomial

$$x^T = x_{i_1} x_{i_2} \cdots x_{i_n},$$

where $\{i_1, i_2, \ldots, i_n\}$ is a listing of all the entries in T without the bars. The *Schur Q-function* is indexed by λ and we define it by

$$Q_\lambda(x) = \sum_T x^T,$$

where the sum is over all semistandard shifted Young tableaux of shape λ. The Schur P-function is defined by

$$P_\lambda(x) = 2^{-l(\lambda)} Q_\lambda(x),$$

where $l(\lambda)$ is the number of rows in a tableau of shape λ. The Schur P-functions form a vector space basis for a subalgebra of the algebra of symmetric functions. For more details, see [12, 14, 19].

3. Kraśkiewicz insertion

We begin this section by giving the background of the Kraśkiewicz insertion algorithm. The description of the algorithm and the B-Coxeter–Knuth relations are found in [10]. We state some results without proof and refer the reader to [10] or [11].

Definition 3.1. Let P be a shifted tableau with l rows and let P_i be the sequence of entries in the ith row of P, reading from left to right. If

1. $\pi_P = P_l P_{l-1} \cdots P_2 P_1$ is a reduced word of w, and
2. P_i is a unimodal subsequence of maximum length in $P_l P_{l-1} \cdots P_{i+1} P_i$,

then P is called a *standard decomposition tableau* of w. We denote the set of such tableaux by $\mathrm{SDT}(w)$. The word π_P is called the *reading word* of P.

Let $w \in B_n$ and $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$. The insertion algorithm gives a map

$$a_1 a_2 \cdots a_m \xrightarrow{\mathrm{K}} (P, Q),$$

from $R(w)$ to pairs of tableaux (P, Q) where $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau. In the literature, P is called the *insertion tableau* and Q is called the *recording tableau*.

We first construct a sequence of pairs of tableaux

$$(\emptyset, \emptyset) = (P^{(0)}, Q^{(0)}), (P^{(1)}, Q^{(1)}), \dots, (P^{(m)}, Q^{(m)}) = (P, Q),$$

where $\mathrm{sh}(P^{(i)}) = \mathrm{sh}(Q^{(i)})$ for $i = 0, 1, \dots, m$. Each tableau $P^{(i)}$ is obtained by inserting a_i into $P^{(i-1)}$. We denote this as

$$P^{(i-1)} \leftarrow a_i = P^{(i)}.$$

Note that each row in the insertion tableau is unimodal.

Insertion Algorithm

Input: a_i and $(P^{(i-1)}, Q^{(i-1)})$, *Output*: $(P^{(i)}, Q^{(i)})$.

Step 1: Let $a = a_i$ and R = 1st row of $P^{(i-1)}$.

Step 2: Insert a into R as follows:

- *Case* 0: $R = \emptyset$. If the empty row is the kth row, we write a indented $k-1$ boxes away from the left margin. This new tableau is $P^{(i)}$. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.
- *Case* 1: Ra is unimodal. Append a to R and let $P^{(i)}$ be this new tableau. To get $Q^{(i)}$, we add i to $Q^{(i-1)}$ so that $P^{(i)}$ and $Q^{(i)}$ have the same shape. Stop.
- *Case* 2: Ra is not unimodal. Some numbers in the increasing part of R is greater than a. Let b be the smallest number in $R\uparrow$ bigger than or equal to a.
  - *Case* 2.0: $a = 0$ and R contains 101 as a subsequence. We leave R unchanged and go to Step 2 with $a = 0$ and R equal to the next row.
  - *Case* 2.1.1: $b \neq a$. We put a in b's position and let $c = b$.
  - *Case* 2.1.2: $b = a$. We leave the increasing part $R\uparrow$ unchanged and let $c = a + 1$.

We insert c into the decreasing part $R\downarrow$. Let d be the biggest number in $R\downarrow$ which is smaller than or equal to c. This number always exists because the minimum of a unimodal sequence is in its decreasing part.

— *Case* 2.1.3: $d \neq c$. We put c in d's place and let $a' = d$.

— *Case* 2.1.4: $d = c$. We leave $R\downarrow$ unchanged and let $a' = c - 1$.

Step 3: Repeat Step 2 with $a = a'$ and R equal to the next row.

Example: Let $a = 01321 \in R(42\bar{1}3)$.

$$P^{(1)} = \boxed{0} \qquad Q^{(1)} = \boxed{1}$$

$$P^{(2)} = \boxed{0}\boxed{1} \qquad Q^{(2)} = \boxed{1}\boxed{2}$$

$$P^{(3)} = \boxed{0}\boxed{1}\boxed{3} \qquad Q^{(3)} = \boxed{1}\boxed{2}\boxed{3}$$

In the above, we only use the Case 0 and Case 1 of the insertion algorithm. In the next step, we have to use the other cases. For clarity, we use the symbol '|' to separate the unimodal sequence into decreasing and increasing parts.

$$P^{(3)} \leftarrow 1 = \begin{array}{|c||c|c|}\hline 0 & 1 & 3 \\ \hline\end{array} \leftarrow 2$$

$$\overset{\overset{3}{\leftarrow}}{} \quad = \begin{array}{|c||c|c|}\hline 0 & 1 & 2 \\ \hline\end{array}$$

$$= \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & \leftarrow 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$$= \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} = P^{(4)}$$

and

$$Q^{(4)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$$P^{(4)} \leftarrow 1 = \begin{array}{|c|c||c|}\hline 3 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array} \leftarrow 1$$

$$\overset{\overset{2}{\leftarrow}}{} \quad \begin{array}{|c|c||c|}\hline 3 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$$

$$= \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & 1 \\ \cline{2-3}\end{array} = P^{(5)}$$

and

$$Q^{(5)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3}\end{array}.$$

We state below the result proved by Kraśkiewicz.

Theorem 3.2 (Kraśkiewicz [10, Theorem 5.2]). *Kraśkiewicz insertion is a bijection between $R(w)$ and pairs of tableaux (P, Q) where $P \in \mathrm{SDT}(w)$ and Q is a standard shifted Young tableau of the same shape as P.*

With this result, it follows immediately that:

Corollary 3.3 (Kraśkiewicz [10, Section 6]). *Let $w \in B_n$:*

$$|R(w)| = \sum_{P \in \mathrm{SDT}(w)} g^{\mathrm{sh}(P)},$$

where g^{λ} is the number of standard shifted Young tableaux of shape λ.

In [10], a number of the results and proofs depend on what we call the B-Coxeter–Knuth relations. We translate them into our notation. If $\boldsymbol{a} = a_1 a_2 \cdots a_m$, define $\boldsymbol{a}^r = a_m \cdots a_2 a_1$.

Definition 3.4. Let $a, b \in R(w)$ for some $w \in B_n$. Suppose $\boldsymbol{a} = \boldsymbol{cxd}$ and $\boldsymbol{b} = \boldsymbol{cyd}$. If $\boldsymbol{x} \sim \boldsymbol{y}$ or $\boldsymbol{x}^r \sim \boldsymbol{y}^r$ appears in the list below, we say that $\boldsymbol{a}$ is elementary *B-Coxeter–Knuth related to* $\boldsymbol{b}$:

$$0101 \sim 1010 \tag{1}$$

$$abb+1b \sim ab+1bb+1 \tag{2}$$

$$bab+1b \sim bb+1ab \tag{3}$$

$$aa+1ba \sim aa+1ab, \quad a+1<b \tag{4}$$

$$a+1aba+1 \sim a+1baa+1, \quad a+1<b \tag{5}$$

$$abdc \sim adbc \tag{6}$$

$$acdb \sim acbd \tag{7}$$

$$adcb \sim dacb \tag{8}$$

$$badc \sim bdac \tag{9}$$

Let $\boldsymbol{e}, \boldsymbol{f} \in R(w)$. If there exists a sequence of reduced words $\boldsymbol{e} = \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k = \boldsymbol{f}$ such that $\boldsymbol{a}_i$ is elementary B-Coxeter–Knuth related to $\boldsymbol{a}_{i+1}$, we say that $\boldsymbol{e}$ and f are *B-Coxeter–Knuth related.* We denote this by $\boldsymbol{e} \sim \boldsymbol{f}$.

This set of relations also appeared in [8, Table 3]. They are obtained by considering promotion sequences there.

The importance of these relations lies in the following:

Theorem 3.5. *Two reduced words of w are B-Coxeter–Knuth related iff they have the same insertion tableau P.*

This is an analogue of [4, Theorem 6.24] for Edelman–Greene insertion. The proof can be found in [10, Theorem 5.2, 11].

An immediate and simple corollary is:

Corollary 3.6 (Lam [11, Corollary 1.19]). *Let P be a standard decomposition tableau. Then*

$$\pi_P \xrightarrow{\mathrm{K}} (P, Q).$$

Using the B-Coxeter–Knuth relation, Kraśkiewicz also proved that

Lemma 3.7 (Kraśkiewicz [11, Lemma 4.8]). *Let $\boldsymbol{a} \in R(w)$ and*

$$\boldsymbol{a} \xrightarrow{\mathrm{K}} (P, Q).$$

Then the length of the longest unimodal subsequence in $\boldsymbol{a}$ is λ_1, the length of the first row of P.

3.1. Properties of Kraśkiewicz insertion

In this subsection, we study in more detail the behaviour of Kraśkiewicz insertion. This enables us to express the B_n Stanley symmetric functions G_w in terms of Schur P-functions.

Let $\boldsymbol{a} \in R(w)$ and suppose $a_i a_{i+1} \cdots a_k$ is unimodal. Our aim is to show that after applying Kraśkiewicz insertion to $\boldsymbol{a}$, the entries $i, i+1, \ldots, k$ in the recording tableau Q satisfy the properties described below.

Definition 3.8. Suppose the entries $i, i+1, \ldots, k$ appear in Q in the following manner: There exists j such that

1. the entries $i, i+1, \ldots, j$ form a vertical strip,
2. these entries are increasing down the vertical strip,
3. the entries $j, j+1, \ldots, k-1, k$ form a horizontal strip,
4. these entries are increasing from left to right, and
5. each box in the vertical strip is left of those boxes in the horizontal strip that are in the same row.

We say that these entries form a *vee* in Q.

This terminology is borrowed from [2].

Example. In the standard shifted Young tableau below, the entries 3,4,5,6 form a vee:

1	2	3	6
	4	5	

We start by observing the effect of inserting two numbers a, a' into a row R of some standard decomposition tableau. Assume throughout this subsection that Raa' is a reduced word of some signed permutation. We use the notation

$$R \xleftarrow{\text{in}} a = e \xleftarrow{\text{out}} R'$$

to mean that inserting a into R results in e being bumped out and R becomes R' after the insertion. If a is appended to R and no number is bumped out, we write

$$R \xleftarrow{\text{in}} a = Ra.$$

Let us deal with the situation when $a < a'$.

Case 1: Ra is unimodal. Then, Raa' is also unimodal and

$$R \xleftarrow{\text{in}} aa' = Raa'.$$

Case 2: Ra is not unimodal but $R'a'$ is unimodal. Then

$$R \xleftarrow{\text{in}} aa' = e \xleftarrow{\text{out}} R'a'.$$

Case 3: Ra, $R'a'$ are not unimodal. During the insertion of a into R, a bumps c from $R\uparrow$ which in turn bumps e from $R\downarrow$ to the next row. Similarly, a' bumps c' which bumps e'. We assume that $c > a$, $c > e$, $c' > a'$, $c' > e'$ and that c, c', e, e' are distinct. So,

$$\begin{aligned} R \xleftarrow{\text{in}} aa' &= \cdots e \cdots \mid \cdots c \cdots \xleftarrow{\text{in}} aa' \\ &= e \xleftarrow{\text{out}} \cdots c \cdots \mid \cdots a \cdots \xleftarrow{\text{in}} a'. \end{aligned}$$

Since $a' > a$, a' has to bump some number bigger than a. This means that c' lies to the left of a in R'. Similarly, e' lies to the right of c in R', Therefore,

$$R' = \cdots e' \cdots c \cdots a \cdots a' \cdots \Rightarrow R = \cdots e' \cdots e \cdots c \cdots c' \cdots .$$

Clearly, $e < e'$.

Similarly, it can be shown that in all the other cases where e and e' are bumped out by a and a', respectively, $e < e'$ still holds.

If we put all these together, we get:

Lemma 3.9. *Let*

$$a_1 a_2 \cdots a_{m-1} a_m \xrightarrow{\text{K}} (P, Q),$$

where $a_{m-1} < a_m$. *Denote the box in the ith row and jth column of P by* (i, j). *Suppose that in Q, the entries* $m-1$, m *are in boxes* (i, j), $(i' j')$, *respectively. Then* $i \geqslant i'$ *and* $j < j'$.

In other words, $m-1$, m *form a horizontal strip of length two in Q and* $m-1$ *is to the left of m.*

Proof. Let S be the insertion tableau of $a_1 a_2 \cdots a_{m-2}$. We use induction on the number of rows in S. Let S_1 be the first row of S. Following the previous discussion:

In Case 1, both a_{m-1}, a_m get appended to S_1. So, $i = i' = 1$ and $j' = j + 1$.

In Case 2, a_{m-1} bumps some number to the second row and a_m is appended. So, $i > i' = 1$ and $j < j'$.

In the other cases, both a_{m-1}, a_m bump e, e' into the second row and $e < e'$. If S has only one row, then $i = i' = 2$ and $1 = j < j' = 2$.

If S has more than one row, let S' be the resulting standard decomposition tableau after deleting the first row. Then e, e' is to be inserted into S'. Since S' has fewer rows than S and $e < e'$, by the induction hypothesis, $i \geqslant i'$ and $j < j'$. □

Next, we assume that $a > a'$. However, we find that the result is not as nice. For example, if a bumps e out of R and a' bumps e' out of R', then it is possible to have $e < e'$ in some cases and $e > e'$ in other cases. Nonetheless, we still can get some results by extending the case by case analysis to the insertion of $aa'a''$ into R. We summarize them below. The proof is omitted. A proof of the second part can be found in [11, Lemma 2.11].

Lemma 3.10. *Let*

$$a_1 a_2 \cdots a_{m-2} a_{m-1} a_m \xrightarrow{\mathrm{K}} (P, Q).$$

Suppose that in Q, the entries $m-2$, $m-1$, m are in boxes (i,j), (i',j'), (i'',j''), respectively.

1. *If $a_{m-2} > a_{m-1} > a_m$ and $i \geqslant i', j < j'$, then $i' \geqslant i'', j' < j''$.*
2. *If $a_{m-2} < a_{m-1}$ and $a_{m-1} > a_m$, then $i \geqslant i', j < j'$ and $i' > i'', j' \leqslant j''$.*

To rephrase, if $a_{m-2}a_{m-1}a_m$ is a decreasing sequence and the entries $m-2$, $m-1$ form a horizontal strip in Q, then $m-2$, $m-1$, m form a horizontal strip too.

If $a_{m-2}a_{m-1}a_m$ is not unimodal, that is a_{m-1} is bigger than both a_{m-2}, a_m, then $m-2$, $m-1$ form a horizontal strip while $m-1$, m form a vertical strip in Q.

Theorem 3.11. *Let $\boldsymbol{a} = a_1 a_2 \cdots a_m \in R(w)$ and*

$$\boldsymbol{a} \xrightarrow{\mathrm{K}} (P, Q).$$

The subsequence $a_i a_{i+1} \cdots a_k$ is unimodal iff the entries i, $i+1, \ldots, k$ form a vee in Q.

Proof. ($\Rightarrow$) Suppose a_h is the minimum in the unimodal sequence $a_i a_{i+1} \cdots a_k$. From Lemma 3.9, the entries h, $h+1, \ldots, k$ form a horizontal strip in Q. Let $g \geqslant i$ be the biggest index such that in Q, the entries i, $i+1, \ldots, g$ form a vertical strip. Then, $g \leqslant h$, $a_g > a_{g+1} > \cdots > a_h$ and g, $g+1$ form a horizontal strip in Q. From the first part of Lemma 3.10, g, $g+1$, $g+2$ also form a horizontal strip. Repeating this step, we see that $g, g+1, g+2, \ldots, h$ form a horizontal strip in Q. This means that $i, i+1, \ldots, k$ form a vee in Q.

($\Leftarrow$) Suppose, on the contrary, that $a_i a_{i+1} \cdots a_k$ is not unimodal. Then there exists j, $i < j < k$ such that $a_{j-1} < a_j$ and $a_j > a_{j+1}$. From the second part of Lemma 3.10, the entries $j-1$, $j, j+1$ cannot be part of a vee, contradicting the hypothesis. Therefore, $a_i a_{i+1} \cdots a_k$ has to be unimodal. $\square$

Example: The insertion of $\boldsymbol{a} = 01321$ gives the recording tableau

$$Q = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array}\!\!\!\begin{array}{c} \\ \end{array} \quad \begin{array}{c|c|c|} \hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3} \end{array} .$$

The reduced word $\boldsymbol{a}$ contains the unimodal subsequence 321 and the entries 3, 4, 5 form a vee in Q.

3.2. Main result

Now, we are ready to prove the main theorem of this paper.

Theorem 3.12. *For all $w \in B_n$,*

$$G_w(x) = \sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_0(w)} P_{\mathrm{sh}(R)}(x),$$

where $l(R)$ is the number of rows in R and $l_0(w)$ is the number of bars in the 1-line notation of w.

Proof. We first show that

$$2^{l_0(w)} G_w(x) = \sum_{R \in \mathrm{SDT}(w)} Q_{\mathrm{sh}(R)}(x).$$

From Definition 2.3,

$$2^{l_0(w)} G_w(x) = \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{i} \in K(\boldsymbol{a})} 2^{l_0(w)} x_{|i_1|} x_{|i_2|} \cdots x_{|i_m|}.$$

In order to account for the power of 2, we let out $\boldsymbol{a}$-compatible sequences include those that allow i_j to be barred even when $a_j = 0$. Denote this expanded set of $\boldsymbol{a}$-compatible sequences by $K'(\boldsymbol{a})$. Since the number of zeros in a reduced word of w is $l_0(w)$,

$$2^{l_0(w)} G_w(x) = \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{i} \in K'(\boldsymbol{a})} x_{|i_1|} x_{|i_2|} \cdots x_{|i_m|}.$$

We now exhibit a bijection Φ from $\{(\boldsymbol{a}, \boldsymbol{i}) : \boldsymbol{a} \in R(w), \boldsymbol{i} \in K'(\boldsymbol{a})\}$ to $\{(P, T) : P \in \mathrm{SDT}(w),$ T a semistandard shifted Young tableau such that $\mathrm{sh}(T) = \mathrm{sh}(P)\}$.

Step 1: Apply Kraśkiewicz insertion to $\boldsymbol{a}$. Let

$$\boldsymbol{a} \xrightarrow{\mathrm{K}} (P, Q).$$

This P is the standard decomposition tableau that we want.

Step 2: For each $l = 1, 2, \ldots,$ let $i_j i_{j+1} \cdots i_k$ be the subsequence of all numbers in $\boldsymbol{i}$ which is equal to $\bar{l}$ or l. We know that the corresponding subsequence $a_j a_{j+1} \cdots a_k$ is unimodal with minimum, say a_h. From Theorem 3.11, the entries $j, j+1, \ldots, k$ form a vee in Q. Let g be the entry in the lowest, left most box in the vee. We next replace the entries $j, j+1, \ldots, g-1$ by $\bar{l}$ and $g+1, g+2, \ldots, k$ by l. We replace g by $\bar{l}$ if $i_h = \bar{l}$ and by l if $i_h = l$. This produces a semistandard shifted Young tableau T, and we set

$$\Phi(\boldsymbol{a}, \boldsymbol{i}) = (P, T).$$

Given (P, T), we now describe the inverse map by first constructing Q.

Step 1: We replace all the entries in T by distinct numbers $1, 2, \ldots, m$ as follows: Let $l = 1$.

1. For the boxes in T containing $\bar{l}$, we replace them in an increasing order from top to bottom.
2. For the boxes in T containing l, we replace them in an increasing order from left to right.

We repeat the procedure with $l = 2$, then $l = 3$ and so on. This produces a standard shifted Young tableau Q of the same shape as P.

Step 2: Obtain $\boldsymbol{a}$ from (P, Q) by the inverse Kraśkiewicz insertion.

Step 3: For each $l = 1, 2, \ldots,$ let $j, j+1, \ldots, k$ be all the entries in Q that replaced $\bar{l}$ or l. These entries form a vee in Q. Suppose g is the entry of the lowest, leftmost box in the vee. Then, by Theorem 3.11, $a_j a_{j+1} \cdots a_k$ is unimodal with minimum a_h. Set

$$i_j = i_{j+1} = \cdots = i_{h-1} = \bar{l},$$

$$i_{h+1} = i_{h+2} = \cdots = i_k = l,$$

and

$$i_h = \begin{cases} \bar{l} & \text{if } g \text{ replaces } \bar{l}, \\ l & \text{if } g \text{ replaces } l. \end{cases}$$

By construction, $\boldsymbol{i} \in K'(\boldsymbol{a})$ and this clearly produces the inverse map.

It is not difficult to see that the monomial associated with T is $x_{|i_1|} x_{|i_2|} \cdots x_{|i_m|}$. This gives

$$\begin{aligned} 2^{l_0(w)} G_w(x) &= \sum_{\boldsymbol{a} \in R(w)} \sum_{\boldsymbol{i} \in K'(\boldsymbol{a})} x_{|i_1|} x_{|i_2|} \cdots x_{|i_m|} \\ &= \sum_{P \in \mathrm{SDT}(w)} \sum_{\mathrm{sh}(T) = \mathrm{sh}(P)} x^T \\ &= \sum_{P \in \mathrm{SDT}(w)} Q_{\mathrm{sh}(P)}(x). \end{aligned}$$

Therefore,

$$G_w(x) = \sum_{R \in \mathrm{SDT}(w)} 2^{l(R) - l_0(w)} P_{\mathrm{sh}(R)}(x). \qquad \square$$

Note that $l(R) > l_0(w)$ since every row of R can have at most one 0. This gives the following.

Corollary 3.13. *From all $w \in B_n$, $G_w(x)$ is a nonnegative integer linear combination of Schur P-functions..*

Example. Let $w = \bar{3}41\bar{2}$. Let $(\boldsymbol{a}, \boldsymbol{i}) = (10231023,\ 1\bar{2}2\bar{3}\bar{3}336)$. Note that $\boldsymbol{i} \in K'(\boldsymbol{a})$ but $\boldsymbol{i} \notin K(\boldsymbol{a})$. We now construct $(P, T) = \Phi(\boldsymbol{a}, \boldsymbol{i})$ as described in the proof. Applying Kraśkiewicz insertion to $\boldsymbol{a}$, we get

$$(P, Q) = \left(\begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array}\ ,\quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 8 \\ \hline \multicolumn{1}{c|}{} & 5 & 6 & 7 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array} \right)$$

and

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & \bar{2} & 2 & \bar{3} & 6 \\ \hline \multicolumn{1}{c|}{} & 3 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array}\ .$$

With some more work, it can be shown that SDT($\bar{3}41\bar{2}$) contain only

$$\begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & 0 & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 1 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|c|} \hline 3 & 1 & 0 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 1 & 0 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \end{array}\ .$$

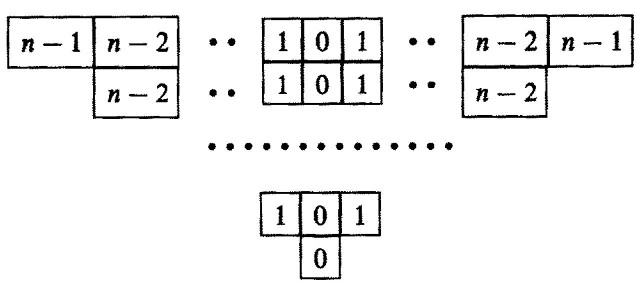
Therefore,

$$G_{\bar{3}41\bar{2}}(x) = 2P_{(5,2,1)}(x) + P_{(5,3)}(x).$$

In [2, Section 3], the definition of the C_n Stanley function is the same as our $G_w(x)$.

Let $w_B = \bar{1}\bar{2} \cdots \bar{n}$. It is the element of longest length in B_n. From [10],

Theorem 3.14 (Kraśkiewicz [10, Corollary 5.3]). SDT (w_B) *contains only the standard decomposition tableau*



Hence,

$$G_{wB} = P_{(2n-1, 2n-3, \ldots, 3, 1)}.$$

Theorem 3.14 can be generalized.

Theorem 3.15. *Let $w \in B_n$ with $w_n = \bar{n}$ and $v = w_1 w_2 \cdots w_{n-1} n$. Then, there exists a bijection between* SDT(v) *and* SDT(w). *Furthermore,*

$$G_w(x) = \sum_{R \in \mathrm{SDT}(v)} 2^{l(R) - l_0(v)} P_{(2n-1,\,\mathrm{sh}(R))}(x),$$

where $(2n-1,\lambda) = (2n-1, \lambda_1, \lambda_2 \cdots)$.

Proof. Let $\boldsymbol{a}$ be any reduced word of w. In order that $w_n = \bar{n}$, $\boldsymbol{a}$ must contain a unimodal subsequence $n-1\ n-2 \cdots 2101 \cdots n-2\ n-1$. So for any $P \in \mathrm{SDT}(w)$, by Lemma 3.7, $|P_1| = 2n-1$. This forces

$$P_1 = n-1\ n-2 \cdots 2101 \cdots n-2\ n-1.$$

Define $\Phi(P)$ to be the tableau after deleting P_1. It can be verified that $\Phi(P) \in \mathrm{SDT}(v)$. This map is clearly a bijection. Since $l(P) = l(\Phi(P)) + 1$, $l_0(w) = l_0(v) + 1$ and $\mathrm{sh}(P) = (2n-1, \mathrm{sh}(\Phi(P))$, the formula for G_w follows. □

Using this, we prove a conjecture by Stembridge [18].

Corollary 3.16. *Let $w \in B_n$ be such that*

$$w_i = \begin{cases} \bar{i} & \textit{for } i = i_1, i_2, \ldots, i_k, \\ i & \textit{otherwise}, \end{cases}$$

where $i_1 < i_2 < \cdots < i_k$ Then,

$$G_w = P_{(2i_k - 1, \ldots, 2i_2 - 1, 2i_1 - 1)}.$$

Proof. Since $w_i = i$ for all $i > i_k$, we can treat w as an element in B_{i_k}. We than apply the previous theorem to get a bijection between SDT(w) and SDT(v) where

$$v_i = \begin{cases} \bar{i} & \text{for } i = i_1, i_2, \ldots, i_{k-1}, \\ i & \text{otherwise.} \end{cases}$$

We repeat the procedure on v and so on. This shows that SDT(w) contains only one tableau with k rows and the jth row is of the form

$$i_j - 1\ i_j - 2 \cdots 101 \cdots i_j - 2 i_j - 1.$$

The formula follows easily. □

4. Further properties of Kraśkiewicz insertion

In this section, we describe some more properties of Kraśkiewicz insertion and relations with other insertion algorithms and operations that appear in the theory of tableaux.

In Section 4.1, we describe some new properties of the standard decomposition tableau. Section 4.2 shows that the Edelman–Greene insertion algorithm can be considered as a special case of the Kraśkiewicz insertion algorithm. Section 4.3 deals with properties of the recording tableau and how it changes when we alter the reduced word. In Section 4.4, we show that Haiman's short promotion sequences for shifted Young tableaux of shape $(2n-1, 2n-3, \ldots, 3, 1)$ may be viewed as inverses to Kraśkiewicz insertion.

4.1. Decreasing parts of insertion tableau

Let P be the insertion tableau of $\boldsymbol{a}$ with shape $(\lambda_1, \lambda_2, \ldots, \lambda_l)$. Recall from Lemma 3.7 that λ_1 is the length of the longest unimodal subsequence in the reduced word $\boldsymbol{a}$. In general, it is difficult to give similar properties for λ_i, $i \neq 1$. However, we are able to say something about the decreasing parts of each row.

Theorem 4.1. *Let P be a standard decomposition tableau and let $P\Downarrow$ be the tableau that is obtained when we delete the increasing parts of each row of P.* Then, *$P\Downarrow$ is a shifted tableau which is strictly decreasing in each row and in each top-left to bottom-right diagonal.*

Remark. If we write $P\Downarrow$ flush left, it becomes an unshifted tableau of shape μ, $\mu_1 > \mu_2 > \cdots > \mu_l$, whose entries are strictly decreasing in rows and columns.

Proof. Obviously, the numbers in each row of $P\Downarrow$ is strictly decreasing. To prove the theorem, it suffices to show for any 2 consecutive rows in P that the decreasing parts have the properties mentioned above. So, let

$$R = \underbrace{a_1 a_2 \cdots a_g}_{R\downarrow} \mid \underbrace{a_{g+1} \cdots a_h}_{R\uparrow},$$

$$S = \underbrace{b_1 b_2 \cdots b_k}_{S\downarrow} \mid \underbrace{b_{k+1} \cdots b_l}_{S\uparrow}$$

be 2 consecutive rows in P with R on top of S.

Note that the subsequence $b_1 b_2 \cdots b_k a_{g+1} \cdots a_h$ or $b_1 b_2 \cdots b_k a_g a_{g+1} \cdots a_h$ is unimodal. By the definition of standard decomposition tableau,

$$k + h - g \leqslant h \ \Rightarrow \ g \geqslant k.$$

We now show by induction on $|S| = l$ that $g > k$ and $a_j > b_j$ for all $j \leqslant k$.

Case: $|S| = 1$. This means that $S = b_1$. Since $b_1 a_1 a_2$ is not a unimodal sequence, $a_1 > b_1$, $a_1 > a_2$ and $g > k = 1$.

Case: $|S| > 1$. Let $j \leqslant k$. Let P' be the standard decomposition tableau that is obtained when we apply the insertion to $b_1 b_2 \cdots b_k b_{k+1} \cdots b_l a_1 \cdots a_j$. Then

$$P' = \begin{array}{cccccccc} \boxed{c_1} & \boxed{c_2} & \bullet\bullet & \boxed{c_{j-1}} & \boxed{b_j} & \bullet\bullet & \boxed{c_{l+1}} \\ & \boxed{b_1} & \boxed{b_2}\,\bullet\bullet & \boxed{b_{j-1}} & & & \end{array}$$

where $c_1 > c_2 > \cdots > c_{j-1} > b_j$. By induction hypothesis, $c_{j-1} > b_{j-1} > b_j$. When we next insert a_{j+1} into P', b_j gets bumped into the second row and it has to be Case 2.1.3 of the insertion algorithm since $c_{j-1} > b_j + 1$. This means that the new entry in the jth box in the first row is strictly bigger than b_j. Hence, $a_j > b_j$ for all $j \leqslant k$.

Suppose, on the contrary, that $g = k$. When we insert a_{k+1}, b_k gets bumped into the second row. Again, by the same argument as above, with j replaced by k, this bumping process involves Case 2.1.3 of the insertion algorithm. But b_k is the smallest element in the first row. This causes the length of the decreasing part of the first row to increase and contradicts $g = k$. Hence, $g > k$. $\square$

The next theorem is an analogue of Lemma 3.7.

Theorem 4.2. *Let* $\boldsymbol{a} \to (P, Q)$ *and* $\mathrm{sh}(P\Downarrow) = \mu$. *Then, any longest strictly decreasing subsequence in* $\boldsymbol{a}$ *has length* μ_1.

Proof. We first prove the theorem when $\boldsymbol{a} = \pi_P$. Let $\boldsymbol{d} = d_1 d_2 \cdots d_k$ be a strictly decreasing subsequence of π_P and let $P_1 = a_1 a_2 \cdots a_{\mu_1} \cdots a_{\lambda_1}$. The subsequence $P_1\downarrow = a_1 a_2 \cdots a_{\mu_1}$ is strictly decreasing and has length μ_1. Suppose $k > \mu_1$. If d_k does not lie in $P_1\uparrow$ and $d_k \neq a_{\mu_1}$, then $\boldsymbol{d} a_{\mu_1}, a_{\mu_1 + 1} \cdots a_{\lambda_1}$ is a unimodal subsequence of π_P of length greater than λ_1. This contradicts Lemma 3.7. If d_k lies in $P_1\uparrow$ or $d_k = a_{\mu_1}$, $d_1 d_2 \cdots d_{k-1} a_{\mu_1} a_{\mu_1+1} \cdots a_{\lambda_1}$ is a unimodal subsequence of π_P of length bigger than λ_1. Similarly, this is a contradiction. Hence, $k \leqslant \mu_1$.

Next, for a reduced word $\boldsymbol{a}$ which is not a reading word, it follows from Theorem 3.5 that $\boldsymbol{a} \sim \pi_P$. It suffices to verify that the length of the longest strictly decreasing subsequence is preserved by the B-Coxeter–Knuth relations. This is analogous to the proof of Lemma 3.7 [10, Lemma 4.8] and we omit this tedious verification. $\square$

Using what we have achieved so far, we give an example of G_w that has an easy description in terms of the Schur P-functions.

Corollary 4.3. *For the signed permutation* $w = \bar{n} \cdots \bar{2}\bar{1}$,

$$G_{\bar{n} \cdots \bar{2}\bar{1}}(x) = P_{(n, n-1, \ldots, 2, 1)}(x).$$

Proof. Let $P \in \mathrm{SDT}(\bar{n} \cdots \bar{2}\bar{1})$. Since $l_0(w) = n$, P must have at least n rows and n '0' entries. Therefore, $|P \Downarrow| \geqslant n(n-1)/2$. But, $l(w) = n(n-1)/2$. This forces $P = P \Downarrow$ and $l(P) = n$. There is only one such P which is

$$\begin{array}{ccccc}
\boxed{n-1} & \boxed{n-2} & \cdots\cdot & \boxed{0} & \\
 & \boxed{n-2} & \cdots\cdot & \boxed{0} & \\
 & & \cdots\cdot & & \\
 & & & \boxed{1}\,\boxed{0} & \\
 & & & \phantom{\boxed{1}}\,\boxed{0}\,. &
\end{array}$$

This gives the formula for $G_{\bar{n}\cdots\bar{2}\bar{1}}(x)$. $\square$

This corollary can be generalized.

Corollary 4.4 (Billey and Haiman [2, Proposition 3.14]). *Let $w = \bar{\lambda}_1 \bar{\lambda}_2 \cdots \bar{\lambda}_l 123 \cdots \hat{\lambda}_l \cdots \hat{\lambda}_{l-1} \cdots \hat{\lambda}_1 \cdots$ where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a shifted shape. Then*, $\mathrm{SDT}(w)$ *contains only*

$$\begin{array}{cccc}
\boxed{\lambda_1 - 1} & \boxed{\lambda_1 - 2} & \cdots\cdots\cdots\cdots\cdots\cdots & \boxed{1}\,\boxed{0} \\
 & \boxed{\lambda_2 - 1} & \cdots\cdots\cdots\cdots\ \boxed{0} & \\
 & & \cdots\cdots\cdots\cdots & \\
 & & \boxed{\lambda_l - 1}\ \cdots\ \boxed{0} &
\end{array}$$

and

$$G_w(x) = P_\lambda(x).$$

Here, $\hat{k}$ means omitting k.

Proof. Since $w_i = i$ for $i > \lambda_1$, we can consider w an an element of B_{λ_1}. Let $\boldsymbol{a}$ be a reduced word of w. First, we note that $\boldsymbol{a}$ must end with a zero. Next, the longest decreasing subsequence in $\boldsymbol{a}$ has to be $\lambda_1 - 1\ \lambda_1 - 2 \cdots 2\,1\,0$. This arises from $w_1 = \bar{\lambda}_1$. By theorem 4.2, for any $P \in \mathrm{SDT}(w)$,

$$P_1 = \lambda_1 - 1\ \ \lambda_1 - 2 \cdots 2\ \ 1\ \ 0.$$

Now, if we let P' be the resulting tableau after deleting P_1, it can be verified that $P' \in \mathrm{SDT}(v)$ where $v = \bar{\lambda}_2 \bar{\lambda}_3 \cdots \bar{\lambda}_l 123 \cdots$. By induction on l, $\mathrm{SDT}(v)$ contains only one tableau P'. It has $l-1$ rows and the jth row has entries

$$\lambda_{j+1} - 1\ \ \lambda_{j+1} - 2 \cdots 1\ \ 0$$

when read from left right. Therefore, $\mathrm{SDT}(w)$ contains only one tableau P of the form given in the theorem and $G_w(x) = P_\lambda(x) = P_\lambda(x)$. $\square$

This was proven using a different method in [2] and it shows that any Schur P-function can be written as a B_n Stanley symmetric function.

4.2. Edelman–Greene insertion

In this subsection, we exhibit a relation between Edelman–Greene insertion and Kraśkiewicz insertion. First, we introduce the notion of a skew shifted Young tableau.

Definition 4.5. Let Q be a standard shifted Young tableau and S be a subset of Q which is also a standard shifted Young tableau. We define the *skew shifted Young tableau* $Q - S$ to be the tableau that remains after deleting all the boxes in Q that are also in S.

The Edelman–Greene insertion algorithm is the S_n analogue of the Kraśkiewicz insertion algorithm. We refer the reader to [4] for a description of Edelman–Greene insertion and its properties. Let $\boldsymbol{a}$ be a reduced word of a permutation w in S_n. The Edelman–Greene insertion algorithm maps $\boldsymbol{a}$ to a pair of tableaux. Denote this by

$$\boldsymbol{a} \stackrel{\mathrm{EG}}{\rightarrow} (\tilde{P}, \tilde{Q}),$$

where $\tilde{P}$ is the insertion tableau and $\tilde{Q}$ is the recording tableau. Note that $\tilde{P}$, $\tilde{Q}$ are unshifted and of the same shape. The set of insertion tableaux that is obtained from all reduced words of w is denoted by $\mathrm{SDT}_S(w)$.

Example. Consider the permutation $v = 2431$ and reduced word 3123.

$$3123 \stackrel{\mathrm{EG}}{\rightarrow} = \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array}\ , \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & \multicolumn{2}{|c}{} \\ \cline{1-1} \end{array} \right)$$

It turns out that $\mathrm{SDT}(v)$ contains only the insertion tableau shown above. Now, let $w_s = 4321$ and then $w_s v = 3124$. If we look at $\mathrm{SDT}(\bar{3}\bar{1}\bar{2}\bar{4})$, it contains only

$$\begin{array}{ccccccc} \boxed{3} & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{1} & \boxed{2} & \boxed{3} \\ & \boxed{2} & \boxed{1} & \boxed{0} & \boxed{2} & & \\ & & \boxed{1} & \boxed{0} & & & \\ & & & \boxed{0} & & & \end{array}$$

Notice that the Edelman–Greene insertion tableau of 2431 appears to the right of the column of zeros in the Kraśkiewicz insertion tableau of $\bar{3}\bar{1}\bar{2}\bar{4}$ except for the entry in the second row. This is not a coincidence. We describe this phenomenon in the theorem below.

Let $w = w_1 w_2 \cdots w_n \in S_n$ and define $\bar{w}$, an element of B_n, by

$$\bar{w} = \bar{w}_1 \bar{w}_2 \cdots \bar{w}_n.$$

Let $w_S = n\, n-1 \cdots 2\, 1$. It is the longest element in S_n. Recall from Corollary 4.3 that $\bar{w}_s = \bar{n} \cdots \bar{2}\bar{1}$ has only one standard decomposition tableau which we denote by U. Let V be the resulting recording tableau when we apply Kraśkiewicz insertion to π_U.

Theorem 4.6. *Let $w \in S_n$. Take a reduced word $\boldsymbol{a} = a_1 a_2 \cdots a_m$ of $w_s w$ and let*

$$\boldsymbol{a} \overset{\mathrm{EG}}{\to} (\tilde{P}, \tilde{Q}),$$

$$\pi_U \boldsymbol{a} \overset{\mathrm{K}}{\to} (P, Q).$$

Then the following holds:

1. $P \in \mathrm{SDT}(\bar{w})$.
2. $P \Downarrow = U$.
3. *The increasing parts of each row of P form an unshifted tableau, denoted by $P \Uparrow$. If we add $i - 1$ to each box in the ith row of $P \Uparrow$, we get $\tilde{P}$.*
4. *If we subtract $n(n-1)/2$ from every box in $Q - V$, we get $\tilde{Q}$.*

Proof. First, note that $\bar{w}_s w_s w = \bar{w}$ and $l(\bar{w}_s) + l(w_s w) = l(\bar{w})$. Therefore,

$$\pi_U \boldsymbol{a} \in R(\bar{w}) \Rightarrow P \in \mathrm{SDT}(\bar{w}).$$

Since $P\Downarrow$ must contain n 0's, it has to be U.

To prove the next two statements, we use induction on m. These are true when $m = 1$. Assume $m > 1$ and let

$$a_1 a_2 \cdots a_{m-1} \overset{\mathrm{EG}}{\to} (\tilde{P}', \tilde{Q}'),$$

$$\pi_U a_1 a_2 \cdots a_{m-1} \overset{\mathrm{K}}{\to} (P', Q').$$

Now compare the Edelman–Greene insertion of a_m into $\tilde{P}'$ and the Kraśkiewicz insertion of a_m into P'. Let the first row of $\tilde{P}'$ be

$$\tilde{P}'_1 = b_{1,1} b_{1,2} \cdots b_{1,\lambda_1}.$$

Suppose a_m bumps some $b_{1,j} > a_m$ during Edelman–Greene insertion:

$$b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{\text{in}}{\leftarrow} a_m = b_{1,j} \overset{\text{out}}{\leftarrow} b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}.$$

Now let us look at the situation for Kraśkiewicz insertion. By induction, we find that

$$P_1' = n-1\ n-2 \cdots 1\ 0 | b_{1,1} b_{1,2} \cdots b_{1,\lambda_1}$$

and

$$n-1\ n-2 \cdots 1\ 0 | b_{1,1} b_{1,2} \cdots b_{1,j} \cdots b_{1,\lambda_1} \overset{\text{in}}{\leftarrow} a_m$$

$$= n-1\ n-2 \cdots 1\ 0 \overset{\overset{b_{1,j}}{\leftarrow}}{|} b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$

$$= b_{1,j} - 1 \overset{\text{out}}{\leftarrow} n-1\ n-2 \cdots 1\ 0 | b_{1,1} b_{1,2} \cdots a_m \cdots b_{1,\lambda_1}$$

So the number to be inserted into P_2' is less than the number to be inserted into $\tilde{P}_2'$. Also, note that $P_1 \uparrow = \tilde{P}_1$. In a similar fashion, it can be shown that this is still true for other types of bumpings. By repeating the process for all the other rows, it can be shown that the two different insertions of a_m end up in the same row such that $\tilde{P}$ and $P \Uparrow$ have the same unshifted shape. This means that $Q - V$ is just $\tilde{Q}$ with all its entries increased by $n(n-1)/2$. But as the insertion gets to a lower row, the difference between the numbers to be inserted into the next row under Edelman–Greene insertion and under Kraśkiewicz insertion gets bigger. This causes each entry in $\tilde{P}_i$ to be $i-1$ bigger than the corresponding entry in $(P \Uparrow)_i$ as stated in the theorem. $\square$

So, given $P \in \mathrm{SDT}(\bar{w})$, if we define $\Psi(P)$ to be the unshifted tableau that is obtained by adding $i-1$ to each box in the ith row of $P \Uparrow$ for all rows, then Ψ is a map from $\mathrm{SDT}(\bar{w})$ and $\mathrm{SDT}_s(w_s w)$. It is not difficult to see that it is a bijection. Using this, we prove a conjecture by Stembridge [18]. This has also been proved using a different method in [2, Eq. (3.15)].

Theorem 4.7. *Let* $w \in S_n$. These exists a bijection from $\mathrm{SDT}(w)$ *to* $\mathrm{SDT}_s(w_s w)$. *Furthermore,*

$$G_{\bar{w}}(x) = \sum_{\tilde{P} \in \mathrm{SDT}_s(w_s w)} P_{\delta_n + \mathrm{sh}(\tilde{P})}(x),$$

where $\delta_n = (n, n-1, \ldots, 2, 1)$.

Proof. The first statement follows from the previous theorem and the bijection constructed above. The formula follows from $\mathrm{sh}(P) = \delta_n + \mathrm{sh}(\Psi(P))$ and $l_0(\bar{w}) = l(P) = n$. $\square$

4.3. Properties of the recording tableau

In this subsection, we explore the properties of the recording tableau obtained from Kraśkiewicz insertion. In the first part, we characterize the recording tableaux for π_P and $(\pi_P)^r$. This enables us to show that $G_w(x) = G_{w^{-1}}(x)$.

The second part deals with jeu de taquin and evacuation [16]. These can also be applied to a standard shifted Young tableau [7, 8]. We show how it relates to the recording tableau of Kraśkiewicz insertion.

Let Q be a standard shifted Young tableau. We denote by $Q|_j$ the standard shifted Young tableau that is obtained from Q by deleting all the boxes that have entries strictly bigger than j. We say that Q is *row-wise* if $Q_1 = 12 \cdots \lambda_1$, $Q_2 = \lambda_1 + 1\ \lambda_1 + 2 \cdots \lambda_2$ and so on. Also, we say that a skew tableau $Q - S$ is connected if we can get from one box to another using a rook.

Lemma 4.8. *Let $P \in \mathrm{SDT}(w)$ and*

$$\pi_P \xrightarrow{\mathrm{K}} (P, Q),$$

where $\mathrm{sh}(P) = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then for all $i \geqslant 1$,

1. $\mathrm{sh}(Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l}) = (\lambda_i, \lambda_{i+1}, \ldots, \lambda_l)$, *and*
2. $Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l} - Q|_{\lambda_{i+1} + \cdots + \lambda_l}$ *is a connected vee.*

Proof. Let $Q^{(i)} = Q|_{\lambda_i + \lambda_{i+1} + \cdots + \lambda_l}$. Let $P^{(i)}$ be the standard decomposition tableau obtained from P by deleting the first i rows. Then

$$\pi_{P^{(i)}} \xrightarrow{\mathrm{K}} (P^{(i)}, Q^{(i)}).$$

This shows that $\mathrm{sh}(P^{(i)}) = \mathrm{sh}(Q^{(i)}) = (\lambda_i, \lambda_{i+1}, \ldots, \lambda_l)$. It follows that $Q^{(i)} - Q^{(i+1)}$ is connected. Also,

$$P^{(i+1)} \xleftarrow{\mathrm{in}} P_i = P^{(i)}.$$

Since P_i is unimodal, by Theorem 3.11, $Q^{(i)} - Q^{(i+1)}$ is a vee. $\square$

Given $\boldsymbol{a} \in R(w)$, we know that $\boldsymbol{a}^r = a_m a_{m-1} \cdots a_1$ is a reduced word of w^{-1}. The following lemma provides a nice description for the recording tableau of $(\pi_P)^r$.

Lemma 4.9. *Let $P \in \mathrm{SDT}(w)$ and*

$$(\pi_P)^r \xrightarrow{\mathrm{K}} (R, S).$$

Then S is row-wise. Furthermore, $\mathrm{sh}(R) = \mathrm{sh}(S) = \mathrm{s}(P) = \lambda$.

Proof. We prove this by induction on $l = l(P)$. It is obvious when $l = 1$. Assume $l > 1$. By definition

$$\pi_P = P_l P_{l-1} \cdots P_2 P_1 \Rightarrow (\pi_P)^r = P_1^r P_2^r \cdots P_l^r.$$

Let us apply Kraśkiewicz insertion to $(\pi_P)^r$ but restricting ourselves to the changes in the first row. Since P_1^r is a unimodal sequence of longest length in $(\pi_P)^r$, by Lemma 3.7.

$|R_1| = |P_1|$. Therefore, none of the numbers in $P_2^r P_3^r \cdots P_1^r$ is appended to the first row during insertion. Thus,

$$\emptyset \overset{\text{in}}{\leftarrow} P_1^r P_2^r \cdots P_l^r = P_1^r \overset{\text{in}}{\leftarrow} P_2^r \cdots P_l^r$$

$$= A_2 \cdots A_l \overset{\text{out}}{\leftarrow} R_1,$$

where A_i is the sequence of numbers that are bumped out when P_i^r is inserted and $|A_i| = |P_i| = \lambda_i$. Note that each A_i is a unimodal sequence. Let

$$A_l^r \cdots A_3^r A_2^r \xrightarrow{\text{K}} (U, V).$$

We claim that $U_i = A_{i+1}^r$ and $\pi_U = A_l^r \cdots A_3^r A_2^r$. Suppose this in not so. Then, there exists $i > 1$ such that A_i^r is a unimodal subsequence of maximum length in $A_l^r \cdots A_i^r$. Let

$$A_l^r \cdots A_i^r \xrightarrow{\text{K}} (U', V'),$$

where U' has k rows. By theorem 3.5, $A_l^r \cdots A_i^r \sim \pi_U$, and by Lemma 3.7, $|U_1'| > |A_i|$. Now compare the following insertions:

$$\emptyset \overset{\text{in}}{\leftarrow} R_1^r A_l^r A_{l-1}^r \cdots A_i^r = R_1^r \overset{\text{in}}{\leftarrow} A_l^r A_{l-1}^r \cdots A_i^r$$

$$= P_l P_{l-1} \cdots P_i \overset{\text{out}}{\leftarrow} X$$

and

$$\emptyset \overset{\text{in}}{\leftarrow} R_1^r U_k' U_{k-1}' \cdots U_1' = R_1^r \overset{\text{in}}{\leftarrow} U_k' U_{k-1}' \cdots U_1'$$

$$= W_k W_{k-1} \cdots W_1 \overset{\text{out}}{\leftarrow} X,$$

where R_1^r becomes X after the insertion and W_i is the sequence of numbers bumped out of the first row when U_i' is inserted. Note that $|W_1| = |U_1'| > |A_i| = |P_i|$. This shows that the resulting insertion tableaux are different. But $R_1^r U_k' \cdots U_1' \sim R_1^r A_l^r \cdots A_i^r$. This contradicts Theorem 3.5. Hence, $A_l^r A_{l-1}^r \cdots A_2^r = \pi_U$ and $\text{sh}(U) = (\lambda_2, \lambda_3, \ldots, \lambda_l)$. So, by the induction hypothesis,

$$A_2 A_3 \cdots A_l = (\pi_U)^r \xrightarrow{\text{K}} (R', S'),$$

where $\text{sh}(R') = \text{sh}(U) = (\lambda_2, \ldots, \lambda_l)$ and S' is a row-wise standard Young tableau. But R' is just R with the first row deleted. Hence, S is row-wise and $\text{sh}(R) = \text{sh}(S) = \text{sh}(P) = \lambda$. $\square$

In general, it $\boldsymbol{a}$ is not a reading word of some P, we have the following result.

Theorem 4.10. *Let* $\boldsymbol{a} \in R(w)$. *Suppose*

$$\boldsymbol{a} \xrightarrow{\mathrm{K}} (P, Q),$$

$$\boldsymbol{a}^r \xrightarrow{\mathrm{K}} (R, S).$$

Then

$$\mathrm{sh}(Q) = \mathrm{sh}(S).$$

Proof. Clearly, $\boldsymbol{a} \sim \pi_P$ implies $\boldsymbol{a}^r \sim (\pi_P)^r$. This means that

$$(\pi_P)^r \xrightarrow{\mathrm{K}} (R, Q').$$

By the previous lemma, $\mathrm{sh}(S) = \mathrm{sh}(R) = \mathrm{sh}(Q') = \mathrm{sh}(P) = \mathrm{sh}(Q)$. □

The map given by $\boldsymbol{a} \to \boldsymbol{a}^r$ is a bijection between $R(w)$ and $R(w^{-1})$ for all w in B_n. From the previous theorem, this map induces a shape-preserving bijection between $\mathrm{SDT}(w)$ and $\mathrm{SDT}(w^{-1})$ which leads naturally to the next result.

Corollary 4.11. *Let* $w \in B_n$. *Then*

$$G_w(x) = G_{w^{-1}}(x).$$

For the rest of this subsection, we start off with a long but crucial lemma. It shows how the shape of the recording tableau changes when we remove a number from the beginning or the end of the reduced word.

Lemma 4.12. *Let* $\boldsymbol{a} \in R(w)$ *and*

$$a_1 a_2 \cdots a_{m-1} a_m \xrightarrow{\mathrm{K}} (P, Q),$$

$$a_1 a_2 \cdots a_{m-1} \xrightarrow{\mathrm{K}} (P', Q'),$$

$$a_2 \cdots a_{m-1} a_m \xrightarrow{\mathrm{K}} (R, S),$$

$$a_2 \cdots a_{m-1} \xrightarrow{\mathrm{K}} (R', S').$$

Then

$$\begin{array}{ccc} \mathrm{sh}(P') & \subset & \mathrm{sh}(P) \\ \cup & & \cup \\ \mathrm{sh}(R') & \subset & \mathrm{sh}(R) \end{array}.$$

Furthermore, $\mathrm{sh}(R) \neq \mathrm{sh}(P')$ *iff* $\{(p,q), (p',q')\}$ *is not connected, where* $(p,q) = \mathrm{sh}(P) - \mathrm{sh}(P')$ *and* $(p',q') = \mathrm{sh}(P') - \mathrm{sh}(R')$.

Proof. Clearly, $\mathrm{sh}(P') \subset \mathrm{sh}(P)$ and $\mathrm{sh}(R') \subset \mathrm{sh}(R)$. Let

$$a_m a_{m-1} \cdots a_2 a_1 \xrightarrow{\mathrm{K}} (U, V),$$

$$a_m a_{m-1} \cdots a_2 \xrightarrow{\mathrm{K}} (U', V').$$

From Theorem 4.10,

$$\mathrm{sh}(R) = \mathrm{sh}(U') \subset \mathrm{sh}(U) = \mathrm{sh}(P).$$

This gives one inclusion and we can get the last inclusion by the same method. Next, we prove the last statement. Note that $\mathrm{sh}(P) - \mathrm{sh}(R') = \{(p, q), (p', q')\}$.

($\Rightarrow$) By the inclusion of shapes, $\mathrm{sh}(R) \neq \mathrm{sh}(P')$ forces

$$\mathrm{sh}(P) - \mathrm{sh}(R) = (p', q').$$

Since (p, q) and (p', q') are corner boxes of $\mathrm{sh}(P)$, they cannot be connected.

($\Leftarrow$) We begin with some simplifications. Without loss of generality, we can assume $p < p'$. Since all the insertions tableaux are not changed when we replace $a_2 a_3 \cdots a_{m-1}$ by $\pi_{R'}$, we can assume that $a_2 a_3 \cdots a_{m-1} = \pi_{R'} = R'_l R'_{l-1} \cdots R'_2 R'_1$. We then have

$$a_1 R'_l R'_{l-1} \cdots R'_2 R'_1 a_m \xrightarrow{\mathrm{K}} (P, Q),$$

$$a_1 R'_l R'_{l-1} \cdots R'_2 R'_1 \xrightarrow{\mathrm{K}} (P', Q'),$$

$$R'_l R'_{l-1} \cdots R'_2 R'_1 a_m \xrightarrow{\mathrm{K}} (R, S),$$

$$R'_l R'_{l-1} \cdots R'_2 R'_1 \xrightarrow{\mathrm{K}} (R', S'),$$

Compare the insertion tableaux P' and R'. Note that

$$\begin{aligned} & p' > 1 \\ & \quad \Rightarrow |P'_1| = |R'_1| \\ & \quad \Rightarrow P'_1 = R'_1, \end{aligned}$$

since R'_1 is a unimodal subsequence of longest length in $a_1 R'_l R'_{l-1} \cdots R'_2 R'_1$. Next, we proceed to use induction on p.

Case: $p = 1$. So, $|P_1| - |P'_1| + 1$ which implies that $P'_1 a_m$ is unimodal. Therefore,

$$\begin{aligned} & P_1 = P'_1 a_m = R'_1 a_m = R_1 \\ & \quad \Rightarrow |R_1| = |P'_1| + 1 \\ & \quad \Rightarrow \mathrm{sh}(R) \neq \mathrm{sh}(P'). \end{aligned}$$

Case: $p > 1$. So, $|P_1| = |P'_1|$ which implies that $P'_1 a_m$ is not unimodal. Therefore,

$$P'_1 \xleftarrow{\text{in}} a_m = a'_m \xleftarrow{\text{out}} P_1.$$

Now, $R'_1 = P'_1$ and R_1 is also obtained from R'_1 by inserting a_m. Hence, $R_1 = P_1$. We then have

$$a_1 R'_l R'_{l-1} \cdots R'_2 a'_m \xrightarrow{\text{K}} (\hat{P}, \hat{Q}),$$

$$a_1 R'_l R'_{l-1} \cdots R'_2 \xrightarrow{\text{K}} (\hat{P}', \hat{Q}'),$$

$$R'_l R'_{l-1} \cdots R'_2 a'_m \xrightarrow{\text{K}} (\hat{R}, \hat{S}),$$

$$R'_l R'_{l-1} \cdots R'_2 \xrightarrow{\text{K}} (\hat{R}', \hat{S}').$$

where $\hat{P}$, $\hat{P}'$, $\hat{R}$, $\hat{R}'$ are standard decomposition tableaux obtained from the corresponding tableaux by deleting the first row. Since $\mathrm{sh}(\hat{P}) - \mathrm{sh}(\hat{R}')$ is not connected, by induction hypothesis,

$$\mathrm{sh}(\hat{P}') \neq \mathrm{sh}(\hat{R}) \Rightarrow \mathrm{sh}(P') \neq \mathrm{sh}(R). \qquad \square$$

Next, following the notation in [15, Section 3.11], we define *delta operator* Δ and evacuation.

Definition 4.13. Let Q be a standard shifted Young tableau. Define $\Delta(Q)$ to be the resulting tableau after applying the following operations:
1. Remove the entry 1 from, Q.
2. Apply jeu de taquin into this box.
3. Deduct 1 form each of the remaining boxes.

This is essentially the same as [15, Definition 3.11.1]but here, we are applying Δ to a shifted Young tableau. In the notation of [7], $\Delta(Q)$ is the tableau which is obtained by subtracting 1 from every box in $Q(1 \to \infty)$.

Theorem 4.14. *Let $a = a_1 a_2 \cdots a_m \in R(w)$ and suppose*

$$a_1 a_2 \cdots a_m \xrightarrow{\text{K}} (P, Q),$$

$$a_2 \cdots a_m \xrightarrow{\text{K}} (R, S).$$

Then

$$S = \Delta(Q).$$

Proof. Induct on m.

Case: $m = 1$. Trivial.

Case: $m > 1$. Let

$$a_1 a_2 \cdots a_{m-1} \xrightarrow{\text{K}} (P', Q|_{m-1}),$$

$$a_2 \cdots a_{m-1} \xrightarrow{\text{K}} (R', S|_{m-2}).$$

Let $(p,q) = \mathrm{sh}(Q) - \mathrm{sh}(Q|_{m-1})$ and $(p',q') = \mathrm{sh}(Q|_{m-1}) - \mathrm{sh}(S|_{m-2})$. As in Lemma 4.12.

$$\begin{array}{ccc} \mathrm{sh}(Q|_{m-1}) & \subset & \mathrm{sh}(Q) \\ \cup & & \cup \\ \mathrm{sh}(S|_{m-2}) & \subset & \mathrm{sh}(S) \end{array}.$$

By induction hypothesis, $\Delta(Q|_{m-1}) = S|_{m-2}$. Note that (p',q') is the last box to be vacated when we compute $\Delta(Q|_{m-1})$.

Subcase 1: $\mathrm{sh}(S) = \mathrm{sh}(Q|_{m-1})$. From Lemma 4.12, $\{(p,q), (p',q')\}$ is connected. When we apply Δ on Q, the jeu de taquin process can be split into two parts. The first part consists of jeu de taquin moves inside $Q|_{m-1}$ only. They are the same as the jeu de taquin moves used when we compute $\Delta(Q|_{m-1})$. Therefore, the box (p',q') is vacated in this part of the process. The second part slides the entry of box (p,q) into (p',q'). It is not difficult to see then that $\Delta(Q) = S$.

Subcase 2: $\mathrm{sh}(S) \neq \mathrm{sh}(Q|_{m-1})$. From Lemma 4.12, $\{(p,q), (p',q')\}$ is not connected. Using the same reasoning as above, it can be shown that $\Delta(Q) = S$. □

We now define evacuation.

Definition 4.15. Let Q be a standard shifted Young tableau with $|Q| = m$. We define the *evacuation* of Q, denoted by $\mathrm{ev}(Q)$, to be a shifted Young tableau of the same shape as Q such that each box (i,j) has entry $m-k+1$ iff $\mathrm{sh}(\Delta^k(Q))$ and $\mathrm{sh}(\Delta^{k-1}(Q))$ differ in box (i,j).

Again, this is essentially the same as [15, Definition 3.11.1] but we are defining it on shifted Young tableaux instead. An alternate definition can be found in [7, Section 8]. Observe that if Q is of size m, then

$$\mathrm{ev}(\Delta(Q)) = \mathrm{ev}(Q)|_{m-1}.$$

Corollary 4.16. *Let $a \in R(w)$ and*

$$a \xrightarrow{\mathrm{K}} (P,Q),$$

$$a^r \xrightarrow{\mathrm{K}} (R,S).$$

Then,

$$S = \mathrm{ev}(Q).$$

Proof. Let $a = a_1a_2 \cdots a_m$. We use induction on m.

Case: $m = 1$. Trivial.

Case: $m > 1$. Consider Kraśkiewicz insertion of the word $a_2a_3 \cdots a_m$. By Theorem 4.14,

$$a_2a_3 \cdots a_m \xrightarrow{\mathrm{K}} (P', \Delta(Q)).$$

Let $(p, q) = \mathrm{sh}(Q) - \mathrm{sh}(\Delta(Q))$. By definition, box (p, q) in $\mathrm{ev}(Q)$ has entry m. Applying the induction hypothesis, we get

$$a_m a_{m-1} \cdots a_3 a_2 \xrightarrow{\mathrm{K}} (R', S')$$

where $S' = \mathrm{ev}(\Delta(Q)) = \mathrm{ev}(Q)|_{m-1}$. Now,

$$R = R' \overset{\mathrm{in}}{\leftarrow} a_1.$$

This shows that

$$S|_{m-1} = S' = \mathrm{ev}(Q)|_{m-1}.$$

From Theorem 4.10, $\mathrm{sh}(S) = \mathrm{sh}(Q)$ and $\mathrm{sh}(S') = \mathrm{sh}(Q')$. This means that the entry m in S is in the box (p, q) and hence $S = \mathrm{ev}(Q)$. $\square$

Example. Let $\boldsymbol{a} = 214203 \in R(\bar{3}2514)$. Applying Kraśkiewicz insertion to $\boldsymbol{a}$ gives

$$\left(\begin{array}{|c|c|c|c|}\hline 4&2&0&3\\ \hline \multicolumn{1}{c|}{}&2&1&\multicolumn{1}{|c}{}\\ \cline{2-3}\end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline \multicolumn{1}{c|}{}&4&5&\multicolumn{1}{|c}{}\\ \cline{2-3}\end{array}\right)$$

and to 14203 gives

$$\left(\begin{array}{|c|c|c|c|}\hline 4&2&0&3\\ \hline \multicolumn{1}{c|}{}&1&\multicolumn{1}{|c}{}&\\ \cline{2-2}\end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\ \hline \multicolumn{1}{c|}{}&3&\multicolumn{1}{|c}{}&\\ \cline{2-2}\end{array}\right)$$

The reader can check that the second recording tableau can be obtained by applying Δ to the first recording tableau. Applying Kraśkiewicz insertion to $\boldsymbol{a}^r = 302412$ gives

$$\left(\begin{array}{|c|c|c|c|}\hline 4&2&1&2\\ \hline \multicolumn{1}{c|}{}&0&3&\multicolumn{1}{|c}{}\\ \cline{2-3}\end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline \multicolumn{1}{c|}{}&5&6&\multicolumn{1}{|c}{}\\ \cline{2-3}\end{array}\right)$$

Note that $\boldsymbol{a}^r$ is a reduced word of $\bar{3}2514^{-1} = 42\bar{1}53$. It can be verified that third recording tableau can be obtained from the first by evacuation. We have hoped that there is a similar operation on the insertion tableau. However, we have failed to find one.

4.4. *Short promotion sequence*

In [8, Theorem 5.12], the short promotion sequence of a shifted tableau is used to give a bijection between standard shifted Young tableaux of shape $(2n-1, 2n-3, \ldots, 3, 1)$ and reduced words of w_B, the longest element in B_n. The same method produces a bijection between standard Young tableaux of shape $(n-1, n-2, \ldots, 1)$

and reduced words of w_s, the longest element of S_n. In [4, Theorem 7.18], it is shown that the Edelman–Greene insertion algorithm applied to reduced words of w_s is the inverse map of the short promotion sequence for S_n. In this subsection, we prove that Kraśkiewicz insertion, when restricted to $R(w_B)$, is the inverse operation of short promotion sequences for B_n.

Lemma 4.17. *Let $N = n^2$ and $\boldsymbol{a} = a_1a_2 \cdots a_{N-1}a_N \in R(w_B)$. If a_0 is a number such that $a_0a_1a_2 \cdots a_{N-1} \in R(w_B)$, then*

$$a_0 = a_N.$$

Proof.

$$\begin{aligned} &a_0a_1a_2 \cdots a_{N-1} = w_B \\ &\quad\Rightarrow a_0 w_B a_N = w_B \\ &\quad\Rightarrow a_0 = w_B a_N w_B \\ &\quad a_N. \quad \square \end{aligned}$$

The definition of the promotion operator is in [8, Section 4] and the definition for the short promotion sequence is in [8, Section 5]. We restate these definitions here but restricting ourselves to the case of standard shifted Young tableaux of shape $(2n-1, 2n-3, \ldots, 3, 1)$.

Definition 4.18. Let $N = n^2$. Given a standard shifted Young tableau T of shape $(2n-1, 2n-3, \ldots, 3, 1)$, we define the *promotion operator* acting on T to be the tableau obtained after applying the following operations:

1. Delete the largest entry in T.
2. Apply jeu de taquin into the empty box.
3. Put 0 into the box (1, 1).
4. Add 1 to every box.

We denote the resulting tableau by $p(T)$.

The short promotion sequence $\hat{p}(T)$ is the sequence of numbers $r_1r_2 \cdots r_N$ where $r_i = n - k$ if the largest entry of the tableau $p^{N-i}(T)$ is in the kth row.

The promotion operation is almost the inverse of Δ in the sense that

$$\Delta(p(T)) = T\,|_{N-1}.$$

Theorem 4.19. *Let $\boldsymbol{a} \in R(w_B)$ and $\boldsymbol{a} \to (P, Q)$. Then,*

$$\hat{p}(Q) = \boldsymbol{a}.$$

Proof. From Lemma 4.17, we have $a_Na_1a_2 \cdots a_{N-1} \in R(w_B)$. Recall that $\mathrm{SDT}(w_B)$ contains only one tableau. So,

$$a_Na_1a_2 \cdots a_{N-1} \xrightarrow{\mathrm{K}} (P, S).$$

But, by Theorem 4.14, the recording tableau for $a_1 a_2 \cdots a_{N-1}$ is

$$Q|_{N-1} = \Delta(S).$$

By the previous remarks,

$$\Delta(p(Q)) = Q|_{N-1} = \Delta(S),$$

and since $\mathrm{sh}(Q) = \mathrm{sh}(S)$,

$$p(Q) = S.$$

This shows that the promotion operator p acting on Q corresponds to moving the last number in the reduced word $\boldsymbol{a}$ to the first position.

Suppose the largest entry N of Q is in the kth row. Note that N has to be in the last box of that row. By reversing the Kraśkiewicz insertion algorithm, it can be shown that $a_N = n - k = r_N$. We can then repeat the procedure on $a_N a_1 a_2 \cdots a_{N-1}$ to show that $r_{N-1} = a_{N-1}$ and so on. Therefore,

$$\hat{p}(Q) = \boldsymbol{a}. \qquad \square$$

5. Open problems

There are a number of open problems in this area that we would like to explore.

1. Let $\boldsymbol{a} \in B_n$. We know that

$$\boldsymbol{a} \xrightarrow{\mathrm{K}} (P, Q),$$

$$\boldsymbol{a}^r \xrightarrow{\mathrm{K}} (R, \mathrm{ev}(Q)).$$

Is there a nice description of the map from P to R directly?

2. Is there a theory of shifted balanced tableaux? This theory for S_n is well explored in [4]. But our attempts to find a suitable shifted analogue have been futile so far.

3. Is there an analogue of the jeu de taquin here? The jeu de taquin is another way of looking at the Robinson–Schensted algorithm. There are analogues of jeu de taquin for Edelman–Greene insertion and Sagan–Worley insertion (see [19, 14]). It is natural to ask if there is one for Kraśkiewicz insertion.

4. Can Haiman's shifted mixed insertion [7] be modified to give an insertion algorithm that is equivalent to Kraśkiewicz insertion?

5. What are the conditions when SDT(w) contains exactly one tableau? The analogue of this question for Edelman–Greene insertion has been answered [17, Theorem 4.1; 4, Theorem 8.1]. The 2143-avoiding permutations are the only permutations v such that $\mathrm{SDT}_s(v)$ contains only one insertion tableau. The data for B_4 and B_5 suggest the following conjecture.

Conjecture 5.1. Let $w \in B_n$. SDT(w) contains only one tableau iff w avoids the following:

$$\begin{array}{cccc}
321 & \bar{1}32 & 2\bar{3}41 & 3\bar{4}1\bar{2} \\
2143 & \bar{2}31 & \bar{4}1\bar{2}3 & 3\bar{4}\bar{1}\bar{2} \\
2413 & \bar{3}21 & \bar{4}\bar{1}\bar{2}3 & \bar{3}\bar{4}1\bar{2} \\
3142 & \bar{3}2\bar{1} & \bar{2}\bar{3}4\bar{1} & \bar{3}\bar{4}\bar{1}\bar{2} \\
3\bar{1}2 & 32\bar{1} & &
\end{array}$$

We are able to prove this for $w \in B_n, l_0(w) = 0, n-1, n$.

6. What are the conditions for $2^{l_0(w)}G_w = Q_{\lambda/\mu}$ is a skew Schur Q-function? This is an extension of the previous question. Some results have been obtained (see [5, 18]).

7. Extend this theory to D_n. Most of the work in this paper has been generalized to D_n, the group of signed permutations with even number of signs [11].

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